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Interpolation and Approximation from Convex Sets

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Let X be a topological vector space, $Y=\mathbb{R}^n$, $n\in\mathbb{N}$, A be a continuous linear map from X to Y, $C\subset X$, B be a convex set dense in C, and $d\in Y$ be a data point. We derive conditions which guarantee that the set $B\cap A^{-1}(d)$ is nonempty and dense in $C\cap A^{-1}(d)$. Some applications to shape preserving interpolation and approximation are described. © 1998 Academic Press

1. INTRODUCTION

A typical framework suitable for studying shape preserving interpolation and approximation can be described as follows. Let X be a Banach space and let A be a linear map from X to $Y=\mathbb{R}^n$, $n\in\mathbb{N}$. If d is a vector in Y, called a *data point*, then the *unconstrained interpolation* problem associated with the spaces X, Y and the operator A can be formulated as:

$$\text{Find } x\in X \text{ such that } Ax=d. \tag{1}$$

Usually, more than one solution exists, in which case one seeks a "best" solution based on predetermined criteria. For example, if $\|\cdot\|$ is a (semi) norm on X, then an element $x_0\in X$ is sought such that

$$\|x_0\| = \min_{\substack{x\in X\\ Ax=d}} \|x\|. \tag{2}$$

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A popular example of this *unconstrained variational problem* represent the so-called *thin plate splines* introduced by Duchon in 1976 [11].

Unconstrained interpolation does not always provide satisfactory results. For instance, in the case of this plate splines, if the data values d are positive (that is, if each of the components of d is positive), it is natural to require that the interpolant be nonnegative. However, it is known that general solutions to (2) do not satisfy this requirement [32]. Therefore, additional shape constraints are imposed. This means that if C is a subset of X, we seek a solution to the *constrained interpolation* problem

$$\text{Find } x \in C \text{ such that } Ax = d. \tag{3}$$

Frequently, the set C is a *closed convex cone*, e.g., a cone of nonnegative, monotone, or convex functions, or the intersection of a number of shifts of such cones. In case there are many solutions to (3), a possibility is to seek a solution of the *constrained variational problem*

$$\|x_0\| = \min_{\substack{x \in C \\ Ax = d}} \|x\|. \tag{4}$$

In applications, a solution to (3) or (4) is often considered under the additional condition that it belongs to a subspace S of X.

Clearly, a necessary condition for the existence of a solution to (4) is that the data point d be *admissible* [5], i.e.,

$$C \cap A^{-1}(d) \neq \varnothing, \tag{5}$$

where

$$A^{-1}(d) := \{x \in X : Ax = d\}.$$

In order to formulate the objectives of this paper, we make the following assumptions. The symbol X will denote a real topological vector space. In the applications we have in mind, X is usually an infinite dimensional function space, e.g., a space of continuous functions, or a Sobolev space. A will stand for a continuous linear map from X to a finite dimensional space $Y = \mathbb{R}^n$, $n \in \mathbb{N}$ (equipped with the usual topology). Let $d \in Y$. Since $\{d\}$ is closed in Y, the set $A^{-1}(d)$ is a closed affine subspace of X. C will denote a nonvoid set in X defining the desired shape properties. Finally,

$$A[C] := \{Ax : x \in C\} \subset Y$$

will designate the set of all admissible data points, called the *data set*.

With this notation, a shape preserving interpolation operator I: $A[C] \to C$ can be viewed as a selection for the set-valued mapping $A[C] \to 2^C$: $d \mapsto C \cap A^{-1}(d)$. This selection is usually based on a minimization of a suitable (quadratic) functional, such as (4). We point out that this problem is closely related to the so-called *variational inequalities* [15, 21]. Special cases have been analyzed, e.g., in [5, 6, 9, 23, 33], where, among other things, conditions have been derived guaranteeing the existence of a solution.

A typical assumption considered in the literature on shape preserving interpolation is the admissibility of the data. The question of the characterization of the admissibility, which we address here, seems to have attracted less attention. In particular, our aim in this paper is to study the problem of the existence of shape preserving interpolants in the case where the set C is replaced by a convex subset B of C. In this case, the shape preserving interpolation problem is to find an element x from X such that

$$x \in B \cap A^{-1}(d). \tag{6}$$

Usually, the set B is given as the intersection of C with a linear subspace S of X, e.g., a space of (piecewise) polynomials, a space of functions of certain smoothness, or a finite dimensional space as a result of discretization of the problem (4). Alternatively, B could represent a set of elements which are strictly contained in C, in some sense. For example, B could be the set of strictly convex or strongly convex functions (see Section 4.2). If d is an admissible data point, i.e., if d admits interpolation from C, it does not necessarily admit interpolation from B.

In this general setting, the problem has been studied by, among others, Wong [33]. He has utilized a condition according to which an element f of C admits both interpolation and approximation from $C \cap S$, where S is a dense subspace of X, if $d = Af$ is a so-called *Slater point*, i.e., $d \in A[\mathrm{int}(C)]$. However, this condition is fairly restrictive since it requires a nonempty (topological) interior of C.

The above facts lead to the following question: Under what conditions is $B \cap A^{-1}(d)$ nonempty and dense in $C \cap A^{-1}(d)$? In the subsequent sections we derive conditions which are less restrictive then those obtained by Wong. In particular, we will dispose of the restriction that X is a Banach space and that C has a nonempty interior. We also consider the special case that S is finite dimensional. The results are then illustrated in a number of examples of shape preserving interpolation and approximation for which the considerations of Wong do not apply. In fact, many results on shape preserving interpolation and approximation obtained previously by other means follow as simple consequences from the presented unifying approach. A first version of this paper appeared in [25].

2. INTERPOLATION AND SIMULTANEOUS APPROXIMATION FROM DENSE SETS

In this section we derive sufficient conditions which guarantee that elements f in C can be interpolated and simultaneously approximated by elements from the set B, provided B is convex and dense in C.

Since we will be interested in interpolating data from $A[C]$, assuming surjectivity of A, i.e., setting $Y = A[X]$, will not be a restriction for our purposes. For a set $M \subset Y$, the symbol $\mathrm{ri}(M)$ will denote its *relative interior*, i.e., the interior of M in $\mathrm{aff}(M)$, the affine hull of M. In the following we say that B is *dense* in C if $B \subset C \subset \mathrm{cl}(B)$, where $\mathrm{cl}(B)$ is the closure of B.

THEOREM 1. *Let B be a convex and dense subset of C. Then* $\mathrm{ri}(A[C]) = \mathrm{ri}(A[B]) \neq \emptyset$.

Proof. Let M be a set in a topological space and let F be a continuous map from this space to another topological space. Then $\mathrm{cl}(F[M]) = \mathrm{cl}(F[\mathrm{cl}(M)])$. To prove the nontrivial inclusion $\mathrm{cl}(F[\mathrm{cl}(M)]) \subset \mathrm{cl}(F[M])$, first note that $M \subset F^{-1}[\mathrm{cl}(F[M])]$ and hence $\mathrm{cl}(M) \subset F^{-1}[\mathrm{cl}(F[M])]$, by continuity of F. Thus, $F[\mathrm{cl}(M)] \subset \mathrm{cl}(F[M])$ and therefore the claimed inclusion holds. Applying this result to the map A and the sets B, C we obtain $\mathrm{cl}(A[B]) = \mathrm{cl}(A[\mathrm{cl}(B)])$ and $\mathrm{cl}(A[C]) = \mathrm{cl}(A[\mathrm{cl}(C)])$. By our assumption $\mathrm{cl}(B) = \mathrm{cl}(C)$, which gives $\mathrm{cl}(A[B]) = \mathrm{cl}(A[C])$. This implies $\mathrm{ri}(A[B]) \subset \mathrm{ri}(A[C]) \subset \mathrm{ri}(\mathrm{cl}(A[B])$ (see [30, p. 88]). Furthermore, it is well known that $\mathrm{ri}(\mathrm{cl}(L)) = \mathrm{ri}(L)$ whenever L is a convex subset of a finite dimensional space [30, p. 90]. Hence, setting $L = A[B]$ and noting that $A[B]$ is convex, we obtain $\mathrm{ri}(A[B]) = \mathrm{ri}(A[C])$.

The second part of the assertion follows from the fact that every nonempty convex set in a finite dimensional space has a nonempty relative interior [30, p. 89]. ∎

The above theorem can be improved in the sense that elements of C corresponding to interior data points can be simultaneously approximated by elements from B. We first need the following

LEMMA 2. *Let φ be a continuous linear functional on X, and let B be a dense convex subset of $C \subset X$. Suppose $r \in \mathrm{ri}(\varphi[C])$. Then $B \cap \varphi^{-1}(r)$ is dense in $C \cap \varphi^{-1}(r)$.*

Proof. If $\mathrm{ri}(\varphi[C]) = \{r\}$, the lemma holds trivially, since then $\varphi[C] = \{r\}$ and thus $C \subset H := \varphi^{-1}(r)$. Otherwise, if $\varphi[C] \neq \{r\}$, then $\mathrm{ri}(\varphi[C]) = \mathrm{int}(\varphi[C])$. It will be sufficient to prove that for every neighborhood U of $0 \in X$ and every $x \in C \cap H$ we have $(x + U) \cap B \cap H \neq \emptyset$. It is well known that there exists a balanced absorbing neighborhood V

such that $V+V \subset U$. Let $x^+ \in (x+V) \cap \mathrm{cl}(C) \cap H^+$, where $H^+ := \varphi^{-1}[(r, \infty)]$, which is an open set by the continuity of φ. Such an x^+ clearly exists since $r \in \mathrm{int}(\varphi[C])$ and V is absorbing, and since $\mathrm{cl}(C) = \mathrm{cl}(B)$ is convex. Similarly, there exists an $x^- \in (x+V) \cap \mathrm{cl}(C) \cap H^-$, where $H^- := \varphi^{-1}[(-\infty, r)]$. Note that the sets $(x+V) \cap H^+$ and $(x+V) \cap H^-$ are open neighborhoods of $x^+ \in \mathrm{cl}(C)$ and $x^- \in \mathrm{cl}(C)$, respectively. Thus by the density of B in C, there exist two elements b^+ and b^- such that $b^+ \in (x+V) \cap B \cap H^+$ and $b^- \in (x+V) \cap B \cap H^-$. Moreover, there is an element b of the form $b = tb^- + (1-t)\, b^+ = x + t(b^- - x) + (1-t)(b^+ - x)$, $t \in (0, 1)$, such that

$$b \in (x+V+V) \cap B \cap H \subset (x+U) \cap B \cap H.$$

This follows from $\varphi(b^-) < r < \varphi(b^+)$, from the convexity of B, and the fact that V is balanced. Hence, we conclude that $(x+U) \cap B \cap H \neq \emptyset$, which finishes the proof. ∎

THEOREM 3. *Let B be a dense convex subset of C and let $d \in \mathrm{ri}(A[C])$. Then $B \cap A^{-1}(d)$ is dense in $C \cap A^{-1}(d)$.*

Proof. We proceed by induction on n, the dimension of Y. The proof for $n=1$ follows from Lemma 2 by setting $r=d$. As for the induction step, let E be a linear map and φ be a linear functional defined for $x \in X$ by $Ex = (r_1, ..., r_n)$ and $\varphi(x) = r_{n+1}$, respectively, whenever $Ax = (r_1, ..., r_n, r_{n+1}) \in \mathbb{R}^{n+1}$. Clearly, since A is continuous, so are E and φ. Let $d = (d_1, ..., d_n, d_{n+1}) \in \mathbb{R}^{n+1}$ such that $d \in \mathrm{ri}(A[C])$. Hence, also $(d_1, ..., d_n) \in \mathrm{ri}(E[C])$ and $d_{n+1} \in \mathrm{ri}(\varphi[C \cap E^{-1}(d_1, ..., d_n)])$. To see this, it is well known that if L is a nonempty convex set in Y, and F is an affine map on Y, then $\mathrm{ri}(F[L]) = F[\mathrm{ri}(L)]$ [19, p. 107]. This readily implies the first assertion $(d_1, ..., d_n) \in \mathrm{ri}(E[C])$, since $E[C]$ is an image of $A[C]$ under a linear map. To prove the latter, let ϕ be the functional defined by $\phi(r_1, ..., r_n, r_{n+1}) := r_{n+1}$, so that $\varphi = \phi \circ A$. Thus, $\varphi[C \cap E^{-1}(d_1, ..., d_n)] = \phi[A[C \cap A^{-1}[\{(d_1, ..., d_n)\} \times \mathbb{R}]]] = \phi[A[C] \cap (\{(d_1, ..., d_n)\} \times \mathbb{R})]$. The assertion $d_{n+1} \in \mathrm{ri}(\varphi[C \cap E^{-1}(d_1, ..., d_n)])$ is now an immediate consequence of $d \in \mathrm{ri}(A[C])$ and $d \in \mathrm{ri}(\{(d_1, ..., d_n)\} \times \mathbb{R})$, and therefore also $d \in \mathrm{ri}(A[C] \cap (\{(d_1, ..., d_n)\} \times \mathbb{R}))$.

By the induction hypothesis, $B \cap E^{-1}(d_1, ..., d_n)$ is dense in $C \cap E^{-1}(d_1, ..., d_n)$. Applying Lemma 2 to the sets $B \cap E^{-1}(d_1, ..., d_n)$ and $C \cap E^{-1}(d_1, ..., d_n)$, with $r = d_{n+1}$, shows that $B \cap E^{-1}(d_1, ..., d_n) \cap \varphi^{-1}(d_{n+1})$ is dense in $C \cap E^{-1}(d_1, ..., d_n) \cap \varphi^{-1}(d_{n+1})$. This proves the assertion of the theorem, since obviously $E^{-1}(d_1, ..., d_n) \cap \varphi^{-1}(d_{n+1}) = A^{-1}(d)$. ∎

Weaker variants of Theorems 1 and 3 have been obtained by Wong [33]. He stated the results for the so-called *Slater points*, i.e., for points d

for which $d \in A[\mathrm{int}(C)]$ or, equivalently, $\mathrm{int}(C) \cap A^{-1}(d) \neq \emptyset$ [5]. In the following we derive his results from Theorem 3. We shall first prove

LEMMA 4. *Let S be a dense subspace of X and let C be a convex set such that* $\mathrm{int}(C) \neq \emptyset$. *Then $C \cap S$ is dense in C.*

Proof. Let $x \in \mathrm{int}(C)$. Then for all open neighborhoods $U(x)$ of x, $\mathrm{int}(C) \cap U(x)$ is nonempty and open. Hence, by the density of S, $\mathrm{int}(C) \cap U(x) \cap S \neq \emptyset$. This means $\mathrm{int}(C) \cap S$ is dense in $\mathrm{int}(C)$ and thus, by convexity of C, and by the assumption $\mathrm{int}(C) \neq \emptyset$, it is also dense in C [20, p. 59]. ▮

COROLLARY 5 (Wong [33]). *Let S be a dense linear subspace of a Banach space X, $C \subset X$ be a convex set, and let* $\mathrm{int}(C) \cap A^{-1}(d) \neq \emptyset$. *Then $C \cap S \cap A^{-1}(d)$ is dense in $C \cap A^{-1}(d)$.*

Proof. Since X is a Banach space, by the open mapping theorem each Slater point $d \in A[\mathrm{int}(C)]$ is also an interior data point, i.e., $d \in \mathrm{int}(A[C]) = \mathrm{ri}(A[C])$ (cf. [5]), and thus Theorem 3 applies. ▮

We conclude the section with some remarks.

Remark 1. Theorem 3 can we viewed as a generalization of the *Singer–Yamabe Theorem* [20], which is obtained if we set $C = X$ and assume that B is a dense convex subset of X. Our proof of the theorem and the preceding lemma follows closely the proof of the Singer–Yamabe Theorem in [20, p. 49]. A different generalization has been given by Deutsch and Morris [10] who considered a norm-preserving simultaneous interpolation and approximation.

Remark 2. The assertion of Theorem 3 serves as a prerequisite in convergence considerations for finite dimensional approximations in shape preserving interpolation problems. In that context it is necessary that discretizations of $K = C \cap A^{-1}(d)$ become dense in K; cf. [15, 33].

Remark 3. The requirement $\mathrm{int}(C) \neq \emptyset$ used by Wong is often too strong, and in many practical situations is not met. In fact, the examples given in [33] only refer to constraints on function values and not on derivatives. Hence, there the cone of positive functions which has a nonempty interior has been considered. The results of Wong do not apply in cases where C are cones of continuous monotone and/or convex functions, since these have empty interiors in the considered function spaces. However, even if the interior of C is empty, the set C might still admit interpolation from $C \cap S$ (examples of such cases will be discussed later in Section 4). In Theorem 1, a weaker condition has been used, namely that

d should be an interior data point. This turns out to be a natural requirement in other situations. For instance, it has been pointed out in [5] that the condition on the data imposed by Micchelli and Utreras [23] in connection with the problem of existence and uniqueness of solutions to certain constrained variational problems is equivalent to d being an interior data point.

3. SHAPE PRESERVING INTERPOLATION IN FINITE DIMENSIONAL SUBSPACES

In this section we consider the case where the set B is the intersection of a nonempty *closed convex* set $C \subset X$ with a *finite-dimensional* space $S \subset X$. Note that B is a closed convex set. Let $\operatorname{rec}(B)$ be the *recession cone* of B in S, that is,

$$\operatorname{rec}(B) := \{x \in S : x + B \subset B\},$$

which is a closed convex cone [20, p. 34]. Our main result is based on the following

THEOREM 6. *Let S be finite dimensional, $A: S \to Y$ be linear, and $B \subset S$ be nonempty, closed, and convex. If $\operatorname{rec}(B) \cap A^{-1}(0)$ is a linear space, then $A[B]$ is closed.*

Proof. If L and M are closed convex sets in a finite dimensional space such that $\operatorname{rec}(L) \cap \operatorname{rec}(M)$ is a space, then $L - M$ is closed [20, p. 104 and Exercise 2.60]. Setting $L = B$ and $M = A^{-1}(0)$ it follows by the assumptions of the theorem that $B + A^{-1}(0)$ is closed. However, it is known that if A is surjective, then $B + A^{-1}(0)$ is closed if and only if $A[B]$ is [20, p. 142]. On the other hand, if A is not surjective we can make it surjective by considering $A[S]$ instead of Y. ∎

Remark 4. If B is a cone, then $\operatorname{rec}(B) = B$, and hence by Theorem 6, $A[B]$ is closed provided $B \cap A^{-1}(0)$ is a space. In particular, $A[B]$ is closed if $B \cap A^{-1}(0) = \{0\}$ [20, p. 105]. The assumption that $B \cap A^{-1}(0)$ is a space cannot be completely removed. Even in the case where B is a closed convex cone it is possible to construct examples for which the data set $A[B]$ is not closed. For instance, A can be the canonical projection operator from $\mathbb{R}^3$ onto $\mathbb{R}^2$ and B can be the closed conical hull of the set $\{(x, 1, z) \in \mathbb{R}^3 : 0 \leqslant x < 1, z \geqslant 1/(1-x)\}$. Clearly, $A[B]$ is not closed in $\mathbb{R}^2$. Note that in this case $\operatorname{rec}(B) \cap A^{-1}(0) = \{(0, 0, z), z \geqslant 0\}$, which is not a space. More sophisticated examples are given in [5] and [20, p. 142]. If B is a polyhedron, (i.e., the intersection of finitely many closed halfspaces), then $A[B]$ is closed even if $B \cap A^{-1}(0)$ is not a space. This follows from the fact that in a finite dimensional space, every polyhedron has a finite basis [30, p. 46]. In fact, $A[B]$ is closed for a polyhedron in any Banach space S, not necessarily finite dimensional [20, p. 180].

COROLLARY 7. *Let S be finite dimensional, $A: S \to Y$ be linear, and C be a nonempty, closed, and convex set such that* $\operatorname{rec}(B) \cap A^{-1}(0)$ *is a linear space, where $B = C \cap S$. If $A[C] \backslash A[B] \neq \varnothing$, then also* $\operatorname{ri}(A[C]) \backslash A[B] \neq \varnothing$.

Proof. By Theorem 6, $A[B]$ is closed. Moreover, $\operatorname{cl}(A[C]) = \operatorname{cl}(\operatorname{ri}(A[C]))$ [19, p. 105]. Suppose $\operatorname{ri}(A[C]) \backslash A[B] = \varnothing$. Thus $\operatorname{ri}(A[C]) \subset A[B]$. But then also $A[C] \subset \operatorname{cl}(A[C]) = \operatorname{cl}(\operatorname{ri}(A[C])) \subset A[B]$, which is impossible, since $A[C] \backslash A[B] \neq \varnothing$. ∎

Remark 5. Corollary 7 can be reformulated as follows. Whenever there exists an admissible data point which cannot be interpolated by elements from B, then necessarily there also exists an *interior* data point which cannot be interpolated from this set. This fact is illustrated in Theorems 10 and 15 below. However, the assertions of Theorem 6 and Corollary 7 do not generally hold if S is a nonlinear finite dimensional manifold in X. This explains why rational splines, splines with variable knots, exponential splines, and various other types of *nonlinear splines* are usually better suited for shape preserving interpolation. We refer the reader to [16, 29] for an overview and references on shape preserving methods based on such splines. Another example is the algebraic curves and surfaces which can interpolate and approximate convex data [17, 22]. Similarly, this is also the case with the (nonlinear) space of parametric geometrically smooth (G^1) piecewise quadratic planar curves, which can be used for convexity preserving interpolation.

Next, we prove that under an additional assumption the sets $B \cap A^{-1}(d)$, $d \in Y$, are compact. This is often important in connection with the existence of shape preserving interpolants defined as minimizers of certain functionals.

THEOREM 8. *Under the assumptions of Theorem* 6, *let $d \in Y$ be such that $B \cap A^{-1}(d) \neq \varnothing$. Then $B \cap A^{-1}(d)$ is compact if and only if* $\operatorname{rec}(B) \cap A^{-1}(0) = \{0\}$.

Proof. Let L and M be closed convex sets in a finite dimensional space such that $L \cap M \neq \varnothing$. Then $\operatorname{rec}(L \cap M) = \operatorname{rec}(L) \cap \operatorname{rec}(M)$ [19, p. 110]. Also, L is compact if and only if $\operatorname{rec}(L) = \{0\}$ [19, p. 109]. The assertion now follows from $\operatorname{rec}(B \cap A^{-1}(d)) = \operatorname{rec}(B) \cap \operatorname{rec}(A^{-1}(d)) = \operatorname{rec}(B) \cap A^{-1}(0)$. ∎

Remark 6. Theorem 8 generalizes an observation made in [13] that for finite dimensional spaces the set of admissible slopes for monotonicity is compact.

4. EXAMPLES

In the following a number of examples will be given illustrating the ideas of the previous sections. The presented examples are nontrivial in the sense that the set C has empty interior. We first establish some notation. We shall assume that Ω is a subset of $\mathbb{R}^s$, $s \in \mathbb{N}$. $C^k(\Omega)$, $k \in \mathbb{Z}_+$, will denote the space of all functions continuously differentiable in Ω up to order k. $\Pi_k(\Omega) := \{p(x) = \sum_{|\alpha| \leqslant k} p_\alpha x^\alpha;\ \ p_\alpha \in \mathbb{R},\ \ \alpha \in \mathbb{Z}^s_+,\ \ x \in \Omega\}$ and $\Pi(\Omega) := \bigcup_{k \in \mathbb{Z}_+} \Pi_k(\Omega)$ denote the space of s-variate polynomials of total degree $\leqslant k$ and the space of all s-variate polynomials in Ω, respectively. Here we employ the standard multi-index notation, i.e., $x^\alpha := x_1^{\alpha_1} \cdots x_s^{\alpha_s}$, $|\alpha| := \alpha_1 + \cdots + \alpha_s$, for $x = (x_1, ..., x_s) \in \mathbb{R}^s$ and $\alpha = (\alpha_1, ..., \alpha_s) \in \mathbb{Z}^s_+$. E will stand for the space of analytic functions in $\mathbb{R}^s$, i.e., functions which can be extended to the complex space $\mathbb{C}^s$ as entire. Finally, if f is a differentiable s-variate real-valued function of variables $x = (x_1, ..., x_s)$, then f_{x_i} will denote the partial derivative of f with respect to x_i.

4.1. *Interpolation and Approximation by Monotone Polynomials*

A continuous function $f \in C(\Omega)$, with Ω a compact subset of $\mathbb{R}^s$, is *nondecreasing* (*increasing*) if $f(x) \leqslant f(y)$ $(f(x) < f(y))$ whenever $y - x \in \mathbb{R}^s_+ \setminus \{0\}$, $x, y \in \Omega$. The cone of all nondecreasing continuous functions in Ω is denoted by $\operatorname{mon}(C(\Omega))$. Let $a \leqslant x^1 < \cdots < x^n \leqslant b$, $n \in \mathbb{N}$, be real numbers. In the following we assume

$$
\begin{aligned}
&X = C[a, b], \qquad a < b,\\
&C = \operatorname{mon}(C[a, b]),\\
&B = \operatorname{mon}(\Pi[a, b]) = C \cap \Pi[a, b],\\
&Y = \mathbb{R}^n,\\
&A\colon X \to Y, \quad Af := (f(x^1), ..., f(x^n)) \in \mathbb{R}^n, \quad f \in X.
\end{aligned}
$$

The cone of all nondecreasing polynomials, $\operatorname{mon}(\Pi[a, b])$, is dense in $\operatorname{mon}(C[a, b])$. This follows from the well-known fact that Bernstein polynomials of a nondecreasing function are nondecreasing. Note that the interior of the data cone $A[C]$ is nonempty since $\operatorname{int}(A[C]) = \{(f_1, ..., f_n) \in \mathbb{R}^n;$

$f_i < f_{i+1},\ i = 1, ..., n-1\}$. Thus, a direct consequence of Theorems 1 and 3 is the following

THEOREM 9. *Let f be an increasing continuous function on $[a, b]$. Then there exists a polynomial p which is nondecreasing on $[a, b]$ and such that $p(x^i) = f(x^i)$, $i = 1, ..., n$. Moreover, the set of all such interpolating nondecreasing polynomials contains a sequence converging uniformly to f.*

We point out that if the interpolation nodes are fixed, the degree k of the interpolating polynomial cannot be specified in advance. This is because of the following.

THEOREM 10. *Let $n \geqslant 3$, $k \in \mathbb{N}$, and $S = \Pi_k[a, b]$. Then there exists an increasing sequence of data values $f_1 < \cdots < f_n$ such that there is no polynomial $p \in C \cap S = \mathrm{mon}(\Pi_k[a, b])$ such that $p(x^i) = f_i$, $i = 1, ..., n$.*

Proof. Choose $f_1, ..., f_n$ such that $f_1 = f_2 < f_3 < \cdots < f_n$. Obviously, $(f_1, ..., f_n) \in A[C]$ and $(f_1, ..., f_n) \notin A[C \cap S]$, since clearly a nonconstant nondecreasing polynomial cannot assume equal values at two different points. Moreover, $\mathrm{rec}(C \cap S) \cap A^{-1}(0) = \{0\}$. Thus, Corollary 7 applies. ▮

Remark 7. The results of Theorems 9 and 10 are not new. They have been presented here for the sake of illustrating the ideas of the previous two sections. For more information and details about monotonicity preserving polynomial interpolation we refer the reader to [28] and references therein.

Remark 8. The above two theorems also hold true in the multivariate case. The proof of Theorem 9 for the multivariate case follows along the same lines. The essential property which must be assured is the denseness of the cone of all nondecreasing polynomials on Ω in $\mathrm{mon}(C(\Omega))$. This can be shown as follows. Let $[a, b]^s$ be an s-dimensional cube containing Ω. It is known [26] that every nondecreasing continuous function in Ω can be extended as a continuous nondecreasing function on $[a, b]^s$. Thus, it is sufficient to show that $\mathrm{mon}(\Pi([a, b]^s))$ is dense in $\mathrm{mon}(C[a, b]^s))$. However, this again follows from the fact that multivariate (tensor product) Bernstein polynomials are monotonicity preserving.

To extend Theorems 9 and 10, we also need to prove the following: Let $D := \{x^1, ..., x^n\} \subset \Omega$ be a set of distinct data sites. The data $f_1, ..., f_n$ corresponding to the data sites are *nondecreasing*, i.e., whenever $x^j - x^i \in \mathbb{R}^s_+$ then $f_i \leqslant f_j$, $i, j = 1, ..., n$, if and only if there exists a function $f \in \mathrm{mon}(C(\Omega))$ interpolating the data. We only sketch the proof of this observation. Obviously, if $f \in \mathrm{mon}(C(\Omega))$ then the data $f(x^1), ..., f(x^n)$ are nondecreasing by definition. The proof in the opposite direction is based on the following fact: Let $f_1, ..., f_n$ be nondecreasing data and let $x^{n+1} \in \mathbb{R}^s$. Then it is possible to find $f_{n+1} \in \mathbb{R}$ such that all the $n+1$ data are nondecreasing. We can take, e.g., $f_l \leqslant f_{n+1} \leqslant f_u$, where $f_l := \max D_l$, $D_l := \{f_i \mid x^i - x^{n+1} \in \mathbb{R}^s_+, i = 1, ..., n\}$, and $f_u := \min D_u$, $D_u := \{f_i \mid x^{n+1} - x^i \in \mathbb{R}^s_+, \ i = 1, ..., n\}$. If D_l or D_u is empty, we define $f_l := \min\{f_i, \ i = 1, \ ..., \ n\}$ and $f_u := \max\{f_i, i = 1, ..., n\}$, respectively. Next, consider a cube $[a, b]^s$ containing the set Ω and the mesh of points $M := \{a, x_1^1, ..., x_1^n, b\} \times \cdots \times \{a, x_s^1, ..., x_s^n, b\}$. By definition, $D \subset M$. We denote the points in $M \backslash D$ as $x^{n+1}, ..., x^{(n+2)^s}$. By the above observation we can find real values $f_{n+1}, ..., f_{(n+2)^s}$ such that the data $f_1, ..., f_{(n+2)^s}$ are nondecreasing. We define the function f such that in each s-dimensional sub-interval of the cube $[a, b]^s$ corresponding to the partition of $[a, b]^s$ determined by M, f is the s-linear blend of all the data associated with the corner points of the sub-interval. This finishes the proof, since clearly f is continuous and also nondecreasing, on account of the fact that multilinear blends preserve monotonicity.

We conclude that in the multivariate case, in addition to the requirement that $n \geqslant 3$ in Theorem 10, it should also be assumed that there exist at least two data sites x^i, x^j in D such that $x^j - x^i \in \mathbb{R}^s_+$.

4.2. *Interpolation and Approximation by Convex Polynomials*

In the following we consider interpolation and approximation of continuous functions by convex polynomials. We shall assume that Ω is a convex compact subset of $\mathbb{R}^s$, $s \in \mathbb{N}$. A continuous function f defined on Ω is *convex* if for arbitrary distinct points y^i, $i = 0, ..., s$, in Ω it holds that

$$f\left(\sum_{i=0}^{s} \lambda_i y^i\right) \leqslant \sum_{i=0}^{s} \lambda_i f(y^i), \tag{7}$$

whenever $\lambda = (\lambda_0, ..., \lambda_s) \in \Lambda := \{(\lambda_0, ..., \lambda_s) \colon \lambda_i > 0, \sum_{i=0}^s \lambda_i = 1, \ i = 0, ..., s\}$. Convex functions are called *strictly convex* if the inequality in (7) is strict. A C^2 function f is called *strongly convex* in Ω if $u^T H_f(x)\, u > 0$, for every $x \in \Omega$ and every $u \in \mathbb{R}^s \backslash \{0\}$, where $H_f(x)$ is the *Hessian matrix* of f at x. Finally, $\operatorname{conv}(C^k(\Omega))$, $\operatorname{conv}(\Pi_k(\Omega))$, $\operatorname{conv}(\Pi(\Omega))$, and $\operatorname{conv}(E(\Omega))$ denote the convex cones of convex functions on Ω belonging to the respective function spaces $C^k(\Omega)$, Π_k, Π, and E.

In order to be consistent with the notation employed in Section 2, we let

$$X = C(\Omega),$$

$$C = \operatorname{conv}(C(\Omega)),$$

$$B = \operatorname{conv}(\Pi(\Omega)) = C \cap \Pi(\Omega),$$

$$Y = \mathbb{R}^n, \qquad n \in \mathbb{N},$$

$$A \colon X \to Y, \quad Af := (f(x^1), ..., f(x^n)) \in \mathbb{R}^n, \quad f \in X.$$

A description of the data cone $A[C]$ of convex functions has been given, e.g., in [7, 24]. According to this description, the data $\{(x^i, f_i)\}$ are convex if and only if there exists a convex piecewise linear interpolant to these data, i.e., a piecewise linear convex function f such that $f(x^i) = f_i$. This, in turn, is equivalent to the existence of a triangulation (or a simplicial decomposition), called a *convex triangulation*, of the data sites $\{x^i\}$ for which the piecewise linear function f based on this triangulation, interpolating the values f_i, is convex. A necessary prerequisite to applying Theorem 1 is the denseness of B in C. Note that in the present situation C has empty interior, thus Lemma 4 cannot be used. In the univariate case the denseness follows immediately from the well-known fact that Bernstein polynomials of a continuous convex function are convex. The proof in the multivariate case is more involved, since there the Bernstein polynomials are not convexity preserving. For the sake of simplicity, we next restrict ourselves to the bivariate case.

THEOREM 11. *Let $s = 2$. A function $f \in \operatorname{conv}(C(\Omega))$ can be approximated arbitrarily well by convex polynomials in Ω in the sense that f is the $C(\Omega)$ limit of a sequence of polynomials from* $\operatorname{conv}(\Pi(\Omega))$.

Proof. The theorem is a direct consequence of Lemmas 12 and 13 below. ∎

LEMMA 12. *Let $s = 2$ and let $f \in \operatorname{conv}(C(\Omega))$. Then f is the $C(\Omega)$ limit of a sequence of analytic functions from* $\operatorname{conv}(E(\Omega))$.

Proof. Let $D = \{x^1, ..., x^n\}$ be a set of distinct data sites in Ω. The proof is based on the following fact. The function f can be approximated arbitrarily well by a convex piecewise linear interpolant based on the data sites D, if n is sufficiently large and if the data sites become dense in Ω. This follows from the fact that a piecewise linear convex interpolant is the best approximant (in the usual norm of $C(\Omega)$) from all piecewise linear interpolants for a fixed set D [24]. Next, for a given set D and a convex triangulation of D, corresponding to the data values $f|_D$, let

$$f_D(x) := \max_{l \in F_{f,D}} l(x), \qquad x \in \mathbb{R}^2,$$

where $L_{f,D}$ denotes the set of linear functions l with the property that whenever x^{i_0}, x^{i_1}, x^{i_2} are three data sites in D, forming a triangle from the considered convex triangulation, then the linear function l interpolating f at these data sites belongs to $L_{f,D}$. Observe that f_D is convex on $\mathbb{R}^2$, it interpolates f at D, and it is bounded from above by the function

$$C_1 \|x\| + C_2, \qquad x \in \mathbb{R}^2, \tag{8}$$

for some $C_1, C_2 \in \mathbb{R}$. Let us consider the following family of approximations to f_D (cf. [31, p. 153]):

$$f_{D,m}(x) := \int_{\mathbb{R}^2} \sigma_m(x-y)\, f_D(y)\, dy, \qquad x \in \mathbb{R}^s,$$
$$\sigma_m(x) := (m/\sqrt{\pi})^s\, e^{-m^2 \|x\|^2}, \qquad m \in \mathbb{N}.$$

On account of (8) it is possible to show that the functions $f_{D,m}$, $m \in \mathbb{N}$, are analytic in $\mathbb{R}^2$ and converge uniformly to f_D on every compact set i.e., also on Ω.

Therefore, it is enough to prove that $f_{D,m}$ is convex in Ω. Let y^0, y^1, y^2 and λ_0, λ_1, λ_2 be given as in (7). We then have

$$\sum_{i=0}^{2} \lambda_i f_{D,m}(y^i) - f_{D,m}\left(\sum_{i=0}^{2} \lambda_i y^i\right)$$
$$= \int_{\mathbb{R}^2} \sigma_m(y) \left(\sum_{i=0}^{2} \lambda_i f_D(y + y^i) - f_D\left(\sum_{i=0}^{2} \lambda_i (y + y^i)\right)\right) dy. \tag{9}$$

The expression (9) is nonnegative since f_D is convex and since σ_m is nonnegative for all $m \in \mathbb{N}$. Thus $f_{D,m}$ is convex, and hence the assertion follows. ▌

LEMMA 13. *The polynomials from* $\operatorname{conv}(\Pi(\Omega))$ *are dense* (*in* $C^\infty(\Omega)$) *in* $\operatorname{conv}(E(\Omega))$.

Proof. Let $f \in \operatorname{conv}(E(\Omega))$. Since strongly convex functions from $\operatorname{conv}(E(\Omega))$ are dense in $\operatorname{conv}(E(\Omega))$, we can assume that f is strongly convex. The truncated Taylor series expansions of an analytic function and all its derivatives are known to converge to this function and its derivatives uniformly on every compact set. Therefore, the Hessian determinant and all its minors, viewed as analytic functions of x, of the truncated Taylor series of f also converge uniformly on any compact set to the Hessian determinant and its minors of f. However, by the assumption of strong convexity of f, the Hessian matrix is positive definite in Ω, i.e., the Hessian determinant of f and its principal minors are positive (analytic) functions in Ω. This means, however, that the sequence of the Taylor polynomials of f contains a subsequence of convex polynomials (in fact, strongly convex) in Ω. ▌

THEOREM 14. *Let* $s = 1$ *or* 2 *and let* $f \in \operatorname{conv}(C(\Omega))$ *be strictly convex and let* $D = \{x^i\}_{i=1}^n$, $n \in \mathbb{N}$ *be a finite set of distinct data sites in* Ω. *Then there exists a polynomial* $p \in \operatorname{conv}(\Pi(\Omega))$ *which interpolates* f *at* D. *Moreover, the set of all such interpolating convex polynomials contains a sequence converging uniformly to* f.

Proof. Since f is strictly convex, the data vector $d = f|_D$ is an interior data point. Therefore, by the density of B in C, an application of Theorems 1 and 3 finishes the proof. ∎

Just as in the case of monotone polynomials, for convex polynomials it is not possible to specify the degree of the polynomials in advance. In fact, to prove this we do not need to restrict to polynomials, since in the case of convexity preservation a more general assertion holds (cf. Theorem 10).

THEOREM 15. *Let $n > 4$ and let $D = \{x^i\}_{i=1}^n \subset \Omega \subset \mathbb{R}$ be a set of distinct data sites. Moreover, let S be a finite dimensional subspace of $C^1(\Omega)$. Then there exist strictly convex data $\{f_i\}_{i=1}^n$ which do not admit convexity preserving interpolation from S.*

Proof. In this case we have

$$X = C(\Omega),$$

$$C = \mathrm{conv}(C(\Omega)),$$

$$Y = \mathbb{R}^n,$$

$$A: X \to Y, \quad Af := f|_D, \quad f \in X.$$

Assume that the data sites are ordered such that $x^1 < \cdots < x^n$. Obviously, the function $f(x) := (x - x^3)_+ := \max\{0, x - x^3\}$, $x \in \Omega$, satisfies $Af \in A[C] \backslash A[C \cap S]$. Moreover, clearly $\mathrm{rec}(C \cap S) \cap A^{-1}(0) = \{0\}$. By Corollary 7, there exist strictly convex data, i.e., data in $\mathrm{int}(A[C])$, which are admissible for interpolation from C and are inadmissible for interpolation from $C \cap S$. ∎

Remark 9. Using a different technique, Theorem 11 has been proved in [34]. In that paper, some negative results about convexity preserving approximation have also been established.

Remark 10. Recently, the first (interpolation) part of Theorem 14 has been proved by other means in [1] and independently in [2]. Here, we have shown the existence of a convex interpolating *polynomial*, while in the two mentioned papers only the existence of a C^∞ function is guaranteed. In [2] the result has been proved for Hermite (gradient) data that are strictly convex (see [4] for the univariate case). By modifying the definitions of A and Y, this generalization can easily be proved using our approach. Finally, in [2] the existence of a *strictly* convex interpolant has been proved. This refinement is easily deduced by replacing the set $\mathrm{conv}(\Pi(\Omega))$ with the subset of all strictly convex, or even strongly convex, polynomials. This replacement is possible since both of these subsets are dense in $\mathrm{conv}(\Pi(\Omega))$.

Remark 11. There seems to be some ambiguity in the literature on bivariate convex interpolation concerning the characterization of strictly convex data. As mentioned above, it is known that the data are convex if and only if there exists a corresponding convex triangulation. It now makes sense to define two cones of "strictly convex data." (1) The first possibility is to consider the cones of data which are convex with respect to a *fixed* triangulation. The union of the interiors of these cones consists of convex data for which no adjacent faces of the corresponding convex piecewise linear interpolant are coplanar (or no four data points are coplanar). This has been done, e.g., in [1] and [3]. (2) An alternative is to consider the cone $A[C]$, i.e., the set of data points for which there exists a convex triangulation. In this case it is easily shown that the interior of this cone, considered in [2] and [24], consists of all data for which the corresponding convex piecewise linear interpolant has a strictly supporting plane at each data point. It is obvious that strictly convex functions give rise to strictly convex data. Note that this is not true in the first case. We point out here that Lemma 3.1 and its proof in [1] holds for data which are strictly convex in the second sense.

Remark 12. From the proof of Theorem 14 it follows that strictly convex data can be interpolated by a convex C^1 function (in fact, a polynomial). It is not difficult to show that actually there exists a *strictly* convex interpolating C^1 function (see Remark 10). Thus, Theorem 15 implies that for every fixed set of at least five distinct data sites, and a finite dimensional space S of C^1 functions, one can find a strictly convex C^1 function for which there is no convex interpolant from S. Theorem 15 appears in [27], and is proved by different means. There a multivariate version of the theorem has also been given. For the sake of brevity, we have restricted ourselves to the univariate case, although a similar proof based on Corollary 7 can be given in the multivariate case.

4.3. *Positive and Monotone C^2 Spline Interpolation*

In [18] it is shown that it is in general impossible to interpolate strictly positive data with nonnegative cubic C^2 splines (with data at the knots). As the authors of [18] point out, for the data $(0, 0)$, $(1, 0)$, $(2, 0)$, $(3, 1)$, $(4, 0)$, $(5, 0)$, all C^2 cubic interpolants are negative at some point in $[0, 5]$ i.e., none of the splines preserve nonnegativity of these data. In our terminology this means that the data point $d=(0, 0, 0, 1, 0, 0)$ belongs to the set of admissible data points for nonnegative interpolation from the set of continuous nonnegative functions (since the data are nonnegative), but that d does not belong to the set of admissible data points for nonnegative interpolation from the given spline space of C^2 cubic splines with knots 0, 1, 2, 3, 4, 5. Note that d is not an interior data point. However, by

Corollary 7, this means that there also exists an interior data point which does not admit nonnegative interpolation with C^2 cubic splines. A similar negative result can be derived for monotone quartic C^2 splines [18].

4.4. *Interpolation to the Vertices of a Polyhedron*

In [17], the following fact was shown by constructive means. For any bounded convex polyhedron in $\mathbb{R}^3$, there is an infinite number of convex interpolating surfaces through the edges of the polyhedron which are G^2 continuous everywhere except at the vertices. Using our results one can prove the existence of a C^∞ surface passing through the vertices of the polyhedron (but not necessarily through its edges). We only sketch the idea of the proof, which is similar to the one of Theorem 14.

After an appropriate translation, a bounded convex polyhedron can be viewed as a continuous spherical convex function, i.e., a continuous positive function f on Ω, the unit sphere in $\mathbb{R}^3$, which represents a convex star-like surface in $\mathbb{R}^3$. This surface is defined as the set of all points of the form $f(x)\,x$, $x \in \Omega$. Just as in the planar case, the set of spherical convex functions is a convex cone. Thus the above problem can be reformulated as: Let f be a spherical convex function and let $D = \{x^1, ..., x^n\}$ be a set of discrete data sites located on Ω such that the data points $f(x^i)\,x^i$, $i = 1, ..., n$, are strictly convex, i.e., such that they are extreme points of their convex hull. Then there exists a C^∞ convex spherical function, interpolating f at D. In fact, as in Section 4.2, one can show that such interpolating functions can also approximate f. Finally, we note that some constructive methods for smooth convexity preserving interpolation can be found in [12, 14].

4.5. *Interpolation with Constrained Length*

Suppose we want to find a curve $\{(x, f(x)), x \in [a, b]\}$, corresponding to a real-valued function f from a space $S \subset C^1[a, b]$, interpolating one dimensional positional data $f_1, ..., f_n$ at data sites $a = x^1 < \cdots < x^n = b$, $n \geqslant 3$. In addition, we require that the lengths of this curve between each two consecutive data points be equal to some prescribed (compatible) values $\ell_1, ..., \ell_{n-1}$. In [8], it has been stated that this may not always be possible. In particular, if S is finite dimensional, then one can find strictly admissible data values $f_1, ..., f_n, \ell_1, ..., \ell_{n-1}$ for which such an interpolating function f does not exist.

To sketch the proof of this statement, we employ the following notation. Let X be the space of functions continuous on $[a, b]$ and continuously differentiable on each subinterval (x^i, x^{i+1}), $i = 1, ..., n-1$. Fixing $\ell_i > x^{i+1} - x^i$, $i = 1, ..., n-1$, we set

$$C := \left\{ f \in X : \int_{x^i}^{x^{i+1}} \sqrt{1 + f'(x)^2}\, dx \leqslant \ell_i,\ i = 1, ..., n-1 \right\},$$

which is a nonempty and convex set [19, p. 5]. Thus setting $Af := (f(x^1), ..., f(x^n))$, we obtain

$$A[C] = \{ d \in \mathbb{R}^n : \sqrt{(x^{i+1} - x^i)^2 + (d_{i+1} - d_i)^2} \leqslant \ell_i,\ i = 1, ..., n-1 \}.$$

Furthermore, let S be a finite dimensional space of C^1 functions. Then it is not difficult to see that $\operatorname{rec}(C \cap S) \cap A^{-1}(0) = \{0\}$. By Theorem 6 we conclude that $A[C \cap S]$ is closed. Note that in fact this also follows directly from the definition of the set C. The proof now proceeds along the same lines as the proofs of Theorems 10 and 15, using Corollary 7. Namely, choosing the values f_i in such a way that

$$\sqrt{(x^{i+1} - x^i)^2 + (f_{i+1} - f_i)^2} = \ell_i, \qquad i = 1, ..., n-1,$$

it is clear that the only possible interpolant from C is the piecewise linear interpolant. Since the data f_i can be chosen such that this is a non-smooth function, it does not belong to the space S. However, by Corollary 7, one can also find strictly admissible data for which there is no interpolant from $C \cap S$. Clearly, this means there is also no interpolant to these data values satisfying

$$\int_{x^i}^{x^{i+1}} \sqrt{1 + f'(x)^2}\, dx = \ell_i, \qquad i = 1, ..., n-1,$$

which finishes the proof.

The above example is interesting in that the interpolation conditions on the length of the curve have been conveniently included in the definition of the set C rather than the definition of the operator A. Otherwise, it would be necessary to consider a nonlinear operator A, for which our theory, as yet, does not apply. Finally, this approach does not seem to make it possible to prove an analog of Theorems 9 and 14. For this, an extension of the theory to a nonlinear setting is needed.

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